**Math 223 Quiz 2 Solutions – September 26, 2005**

(1) $\mathbf{u} = (1,3,6)$ and $\mathbf{v} = (-2,0,4)$.

$$|3\mathbf{u}| = 3|\mathbf{u}| = 3\sqrt{1+9+36} = 3\sqrt{46}$$

$$|-2\mathbf{v}| = 2|\mathbf{v}| = 2\sqrt{4+16} = 2\sqrt{20} = 4\sqrt{5}$$

So,

$$\begin{aligned} |3\mathbf{u}|\mathbf{v} - |-2\mathbf{v}|\mathbf{u} &= 3\sqrt{46}(-2,0,4) - 4\sqrt{5}(1,3,6) \\ &= (-6\sqrt{46}, 0, 12\sqrt{46}) - (4\sqrt{5}, 12\sqrt{5}, 24\sqrt{5}) \\ &= (-6\sqrt{46} - 4\sqrt{5}, -12\sqrt{5}, 12\sqrt{46} - 24\sqrt{5}) \end{aligned}$$

(2) Simplify first:

$$\begin{aligned} & (\mathbf{u}\times\mathbf{w}) - (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times(2\mathbf{u}+\mathbf{v})) \\ = \ & (\mathbf{u}\times\mathbf{w}) - (\mathbf{u}\times\mathbf{v}) + (\mathbf{u}\times(2\mathbf{u})) + (\mathbf{u}\times\mathbf{v}) \\ = \ & \mathbf{u}\times\mathbf{w} \\ = \ & \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 4 \\ -2 & -3 & 5 \end{vmatrix} \\ = \ & (17, -23, -7) \end{aligned}$$

(3)

$$x = 4\cos t, y = 6\sin t, z = 2\sin t$$

$$\mathbf{T} = (-4\sin t, 6\cos t, 2\cos t)$$

$$\begin{aligned} |\mathbf{T}| &= |(-4\sin t, 6\cos t, 2\cos t)| \\ &= \sqrt{16\sin^2 t + 36\cos^2 t + 4\cos^2 t} \\ &= 2\sqrt{4\sin^2 t + 10\cos^2 t} \\ &= 2\sqrt{4(1-\cos^2 t) + 10\cos^2 t} \\ &= 2\sqrt{4 + 6\cos^2 t} \end{aligned}$$

So,

$$\hat{\mathbf{T}} = \frac{(-2\sin t, 3\cos t, \cos t)}{\sqrt{4+6\cos^2 t}}$$

Compute a normal vector:

$$\begin{aligned}
\mathbf{N} &= \frac{d\hat{\mathbf{T}}}{dt} \\
&= \frac{6\cos t\sin t}{(4+6\cos^2 t)^{\frac{3}{2}}}(-2\sin t, 3\cos t, \cos t) + \frac{(-2\cos t, -3\sin t, -\sin t)}{\sqrt{4+6\cos^2 t}}
\end{aligned}$$

When $t = \frac{\pi}{4}$, we get $(x, y, z) = (2\sqrt{2}, 3\sqrt{2}, \sqrt{2})$. So, after some algebra,

$$\mathbf{N}(\frac{\pi}{4}) = \frac{-4}{7\sqrt{14}}(5, 3, 1)$$

and

$$|\mathbf{N}| = \frac{4\sqrt{35}}{7\sqrt{14}}$$

so that

$$\hat{\mathbf{N}} = \frac{\mathbf{N}}{|\mathbf{N}|} = -\frac{(5,3,1)}{\sqrt{35}}$$

and

$$\hat{\mathbf{T}} = \frac{(-2,3,1)}{\sqrt{14}}$$

$$\begin{aligned}
\hat{\mathbf{B}} &= \hat{\mathbf{T}} \times \hat{\mathbf{N}} \\
&= \frac{1}{\sqrt{35\cdot 14}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 3 & 1 \\ -5 & -3 & -1 \end{vmatrix} \\
&= \frac{(0,-7,21)}{\sqrt{35\cdot 14}} \\
&= \frac{(0,-1,3)}{\sqrt{10}}
\end{aligned}$$

(4)

$$x = e^t\cos t, y = e^t\sin t, z = t$$

Curvature for an arbitrarily parameterized curve is

$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$$

$$\dot{\mathbf{r}} = (e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t, 1)$$

$$\ddot{\mathbf{r}} = 2e^t(-\sin t, \cos t, 0)$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = 2e^t(-\cos t, -\sin t, e^t)$$

So that

$$\begin{aligned}|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}| &= 2e^t\sqrt{\cos^2 t + \sin^2 t + e^{2t}} \\ &= 2e^t\sqrt{1 + e^{2t}}\end{aligned}$$

Also,

$$\begin{aligned}|\dot{\mathbf{r}}|^3 &= (e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + 1)^{\frac{3}{2}} \\ &= (e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + e^{2t}(\sin^2 t + 2\sin t \cos t + \cos^2 t))^{\frac{3}{2}} \\ &= (1 + 2e^{2t})^{\frac{3}{2}}\end{aligned}$$

So that

$$\kappa(t) = \frac{2e^t\sqrt{1+e^{2t}}}{(1+2e^{2t})^{\frac{3}{2}}}$$

and

$$\rho(t) = \frac{1}{\kappa(t)} = \frac{(1+2e^{2t})^{\frac{3}{2}}}{2e^t\sqrt{1+e^{2t}}}$$

(5)

$$\begin{aligned}\mathbf{a}(t) &= (3t^2, t+1, -4t^3) \\ \mathbf{v}(t) &= (t^3, \frac{t^2}{2} + t, -t^4) + \mathbf{v}_0 \\ \mathbf{r}(t) &= \left(\frac{t^4}{4}, \frac{t^3}{6} + \frac{t^2}{2}, -\frac{t^5}{5}\right) + \mathbf{v}_0 t + \mathbf{r}_0\end{aligned}$$

Since the particle starts at rest, $\mathbf{v}_0 = (0, 0, 0)$.
Using $\mathbf{r}_0 = \mathbf{r}(0) = (1, 2, -1)$, the answer is

$$\mathbf{r}(t) = \left(\frac{t^4}{4} + 1, \frac{t^2}{2}(\frac{t}{3} + 1), -(\frac{t^5}{5}) - 1\right)$$